POVRŠI DRUGOG REDA

68. Naći centar i poluprečnik sfere

$$x^2+y^2+z^2-3x+5-4z=0$$

$$x^2-3x+\frac{9}{4}+y^2+z^2-4z+4-4+5-\frac{9}{4}=0$$

$2\cdot\frac{3}{2}x=3x$

$$\left(x-\frac{3}{2}\right)^2+y^2+(z-2)^2=\frac{5}{4}$$

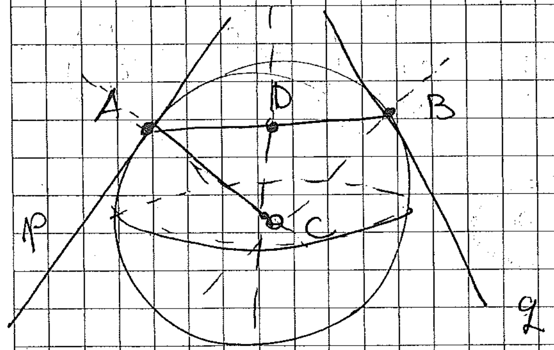
$\rightarrow$ centar $C\left(\frac{3}{2},0,2\right)$; $r=\frac{\sqrt{5}}{2}$

---

69. Naći jednačinu sfere koja dodiruje pravu

$p: \frac{x-1}{3}=\frac{y+4}{6}=\frac{z-6}{4}$ u tački $A(1,-4,6)$

i pravu $q: \frac{x-4}{2}=\frac{y+3}{1}=\frac{z-2}{-6}$ u tački $B(4,-3,2)$



α: ravan koja sadrži A i ortogonalna je na p.

β: ravan koja sadrži B i ortogonalna je na q

γ: ravan koja sadrži D (središte AB) i $\perp$ $\ell(AB)$

$\rightarrow$ u presjeku ovih ravni C

$\alpha\cap\beta\cap\gamma=\{C\}$ $\rightarrow$ centar sfere

$$\underbrace{p: \frac{x-1}{3}=\frac{y+4}{6}=\frac{z-6}{4}}_{\vec{s_p}=(3,6,4)}$$

α: $A(4,-4,6)$

$\vec{n_\alpha}=\lambda\cdot\vec{s_p}$

$n_\alpha=1\cdot(3,6,4)=(3,6,4)$

$\alpha: \quad 3(x-1)+6(y+4)+4(z-6)=0$

$\alpha: \quad 3x+6y+4z-3=0$

→ vektor normale • tačka

$\vec{n}_\alpha = (a, b, c) \qquad \alpha: a\cdot(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$A = (x_0, y_0, z_0)$

$\beta: \quad B(4,-3,2), \quad \vec{n}_\beta = \pi \cdot \vec{s}_\alpha$

$\vec{n}_\beta = (2,1,-6)$

$\beta: \quad 2(x-4)+1(y+3)-6(z-2)=0$

$\beta: \quad 2x+y-6z+7=0$

središte stranice

$\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}$

$D \to \left(\frac{1+4}{2}, \frac{-4-3}{2}, \frac{6+2}{2}\right)$

$D = \left(\frac{5}{2}, -\frac{7}{2}, 4\right)$

$\gamma: \quad D, \quad \vec{n}_\gamma = \pi \cdot \vec{AB}$

$B-A$

$x_b-x_a, y_b-y_a, z_b-z_a$

$\vec{n}_\gamma = (3,1,-4)$

$\gamma: \quad 3\left(x-\frac{5}{2}\right)+1\cdot\left(y+\frac{7}{2}\right)-4\cdot(z-4)=0$

$\gamma: \quad 3x+y-4z+12=0$

$\alpha \cap \beta \cap \gamma = \{C\} \to$ 3 j-ne sa 3 nep.

$C(-5,3,0)$

$R = d(C,A) = d(C,B)$

$R = 11$

→ tražena sfera je:

$(x+5)^2+(y-3)^2+z^2 = 11^2$

121

$\frac{15}{2}+\frac{7}{2} =$

$= \frac{22}{2} + 16 =$

$= \frac{22}{2} + \frac{32}{2} =$

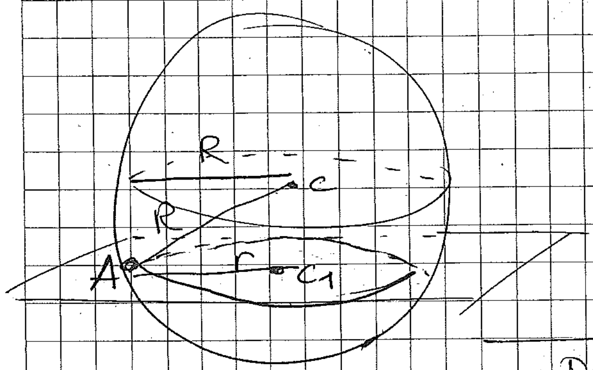
$= \frac{54}{2} =$

10. Naći centar i poluprečnik kruga k;

→ Krug k dat kao presjek sfere i ravni

$$k: \begin{cases} (x-3)^2 + (y+2)^2 + (z-1)^2 = 100 \\ \pi: 2x - 2y - z + 9 = 0 \end{cases}$$

→ provlači se kroz mnoge zadatke



$C(3, -2, 1)$ → centar sfere

$R = 10$, polup sfere

C_1 → centar kruga

r → poluprečnik kruga

Provjerimo da li ravan π prolazi kroz centar sfere

$2 \cdot 3 - 2 \cdot (-2) - 1 + 9 = 0 \rightarrow$

$6 + 4 - 1 + 9 = 0 \rightarrow 18 = 0 \perp$

$C \notin \pi$ → prava koja spaja ova 2 centra uvijek ortogonalna na ravan kruga

π → ravan kruga

PARAMETARSKE J-NE PRAVE

→ postavljamo pravu p tako da:

prava p kroz C i $\perp$ na π

$\vec{s_p} = n \cdot \vec{n_\pi}$

$\vec{s_p} = (2, -2, -1)$

$$\boxed{\begin{gathered} ax + by + cz + d = 0 \\ \vec{n} = (a, b, c) \end{gathered}}$$

$p: \frac{x-3}{2} = \frac{y+2}{-2} = \frac{z-1}{-1}$

$\vec{n}_p = (a,b,c) \qquad p: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

$C \in p = (x_0, y_0, z_0)$

$\rightarrow$ parametarske j-ne

$x = 3+2t$

$y = -2-2t \qquad \longrightarrow C_1(3+2t, -2-2t, 1-t)$

$z = +1-t$

uvrstimo u ravan

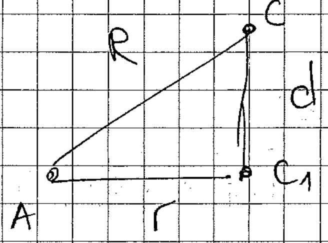
$p \cap \pi = \{C_1\}$

$2\cdot(3+2t) - 2\cdot(-2-2t) - (1-t) + 9 = 0$

$9t + 18 = 0$

$t = -2$

$C_1(-1, 2, 3)$



$r = \sqrt{R^2 - d^2}$

$\cancel{d = (4}$

$d = \sqrt{(3+1)^2 + (-2-2)^2 + (3-1)^2} = \sqrt{16+16+4} = 6$

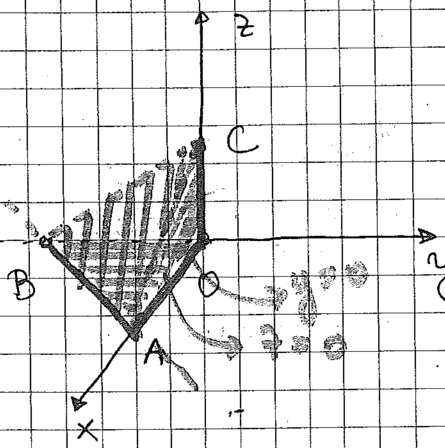
$r = \sqrt{100 - 36} = \sqrt{64} = 8$

$\longrightarrow$ Jna kruga: $(x+1)^2 + (y-2)^2 + (z-3)^2 = 64 = 8^2$

centar upisane sfere

71. Naći jednačinu sfere upisane u tetraedar koji obrazuju ravni:

$\pi_1: x=0$; $\pi_2: y=0$; $\pi_3: z=0$; $\pi_4: 3x-2y+6z=8$



$$\pi_4: 3x-2y+6z=8 \ / \cdot \frac{1}{8}$$

$$\pi_4: \frac{x}{\frac{8}{3}} + \frac{y}{-4} + \frac{z}{\frac{4}{3}} = 1$$

SEGMENTNI OBLIK J-NE RAVNI π_4

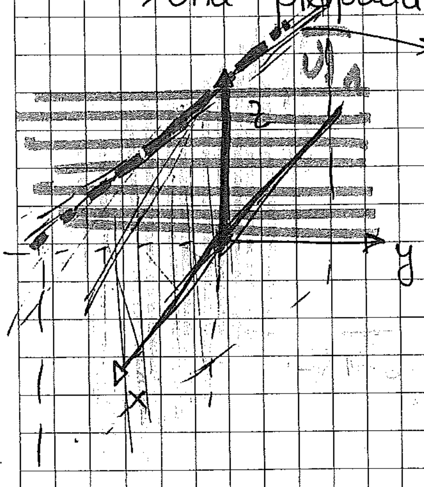
$A\left(\frac{8}{3}, 0, 0\right)$; $B(0, -4, 0)$; $C\left(0, 0, \frac{4}{3}\right)$

tačke u kojima ravan π_4 siječe koor. ose

→ Kad je nešto upisano → simetrale uglova

I Postavimo simetralnu ravan α kroz ivicu OC

→ ona pripada pramenu određenom ravnima π_1 i π_2



→ simetralna ravan α

J-na pramena je:

$$x + \lambda \cdot y = 0$$

$\vec{n}_{\pi_1} = (1,0,0)$

$\vec{n}_{\pi_2} = (0,1,0)$

$$\vec{n}_\alpha = (1, \lambda, 0)$$

$$\sphericalangle(\alpha, \pi_1) = \sphericalangle(\alpha, \pi_2)$$

$$\sphericalangle(\vec{n}_\alpha, \vec{n}_{\pi_1}) = \sphericalangle(\vec{n}_\alpha, \vec{n}_{\pi_2})$$

$$\rightarrow \frac{\overbrace{\vec{n}_\alpha \cdot \vec{n}_1}^{1+0+0}}{\cancel{|\vec{n}_\alpha|}\underbrace{|\vec{n}_1|}_{1}} = \frac{\overbrace{\vec{n}_\alpha \cdot \vec{n}_2}^{0+\lambda+0}}{\cancel{|\vec{n}_\alpha|}\underbrace{|\vec{n}_2|}_{1}} \rightarrow 1 = \lambda$$

$$\boxed{\alpha: x + y = 0}$$

1 0 0

II Postavimo simetralnu ravan β kroz ivicu OA

→ ona pripada pramenu ~~pravih~~ ravni određenih ravnima π_2 i π_3 ;

J-na pramena $\pi_2 + \lambda \pi_3 = 0$

$y + \lambda z = 0$ $\quad \vec{n}_{\pi_2} = (0,1,0)$

$\vec{n}_\beta = (0, 1, \lambda)$ $\quad \vec{n}_{\pi_3} = (0,0,1)$

$\sphericalangle(\beta, \pi_2) = \sphericalangle(\beta, \pi_3) \Rightarrow \sphericalangle(\vec{n}_\beta, \vec{n}_{\pi_2}) = \sphericalangle(\vec{n}_\beta, \vec{n}_{\pi_3})$

$$\frac{\vec{n}_\beta \cdot \vec{n}_{\pi_2}}{|\vec{n}_\beta| \cdot |\vec{n}_{\pi_2}|} = \frac{\vec{n}_\beta \cdot \vec{n}_{\pi_3}}{|\vec{n}_\beta| \cdot |\vec{n}_{\pi_3}|} \qquad \lambda = 1$$

$|\vec{n}_{\pi_2}| \to 1$, $|\vec{n}_{\pi_3}| \to 1$

$$\boxed{\beta : y + z = 0}$$

III Postavimo simetralnu ravan γ kroz ivicu AB

→ ona pripada pramenu određenom ravnima π_3 i π_4

J-na pramena $\cancel{\pi_3 + \lambda \pi_4 = 0}$

$\cancel{z + \lambda(3x - 2y + 6z - 8) = 0}$

LAKŠE: $3x - 2y + 6z - 8 + \lambda z = 0$

$\vec{n}_\gamma = (3, -2, 6+\lambda)$ $\quad \vec{n}_{\pi_4} = (3, -2, 6)$

$\sphericalangle(\gamma, \pi_3) = \sphericalangle(\gamma, \pi_4) \Rightarrow \sphericalangle(\vec{n}_\gamma, \vec{n}_{\pi_3}) = \sphericalangle(\vec{n}_\gamma, \vec{n}_{\pi_4})$

$$\frac{\vec{n}_\gamma \cdot \vec{n}_{\pi_3}}{\cancel{|\vec{n}_\gamma|}\, |\vec{n}_{\pi_3}|} = \frac{\vec{n}_\gamma \cdot \vec{n}_{\pi_4}}{\cancel{|\vec{n}_\gamma|}\, |\vec{n}_{\pi_4}|} \qquad \vec{a} = (a_1, a_2, a_3), \quad \vec{b} = (b_1, b_2, b_3)$$

skrati se

$$\frac{6+\lambda}{1} = \frac{9 + 4 + 6(6+\lambda)}{\sqrt{9+4+36}}$$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$a_1 b_1 + a_2 b_2 + a_3 b_3$

$$6+n = \frac{9+4+6(6+n)}{\sqrt{9+4+36}}$$

$$7(6+n) = 9+4+6(6+n) \rightarrow 6+n = 13$$

$$\boxed{n = 7}$$

$$\gamma : 3x-2y+6z-8+7z=0$$

$$\boxed{\gamma : 3x-2y+13z-8=0}$$

$\alpha \cap \beta \cap \gamma = \{C_1\}$ $C_1 \rightarrow$ centar upisane sfere

$$x+y=0 \Rightarrow y=-x$$

$$y+z=0 \Rightarrow -x+z=0 \Rightarrow z=x$$

$$3x-2y+13z-8=0$$

$$3x+2x+13x-8=0 \rightarrow 18x=8$$

$$\boxed{x = \frac{8}{18} = \frac{4}{9}} \rightarrow y = -\frac{4}{9} \qquad z = \frac{4}{9}$$

$$C_1\left(\frac{4}{9}, -\frac{4}{9}, \frac{4}{9}\right)$$

$$R = d(C_1, \pi_1) = d(C_1, \pi_2) = d(C_1, \pi_3) = d(C_1, \pi_4)$$

$$R = d(C_1, \pi_1) = \frac{\left|\frac{4}{9}\right|}{\sqrt{1}} = \frac{4}{9}$$

$\rightarrow$ J-na sfere:

$$\left(x-\frac{4}{9}\right)^2 + \left(y+\frac{4}{9}\right)^2 + \left(z-\frac{4}{9}\right)^2 = \frac{16}{81}$$

Rastojanje između tačke i ravni

$\rightarrow$ $M(x_0, y_0, z_0)$ $\alpha : ax+by+cz+d=0$

$$d(M,\alpha) = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Rastojanje tačke $\mapsto M$ i prave p određene tačkom $P \in p$ i vektorom pravca $\vec{p}$

$$d = \frac{|\vec{p} \times \overrightarrow{PM}|}{|\vec{p}|}$$

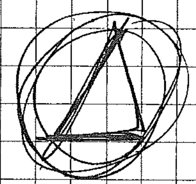
centar opisane sfere

72) Naći jednačinu sfere koja je opisana oko tetraedra čija su tjemena:

$A(2,0,0)$; $B(0,5,0)$; $C(0,0,3)$; $O(0,0,0)$

I Način: Neka je:

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$ jna tražene sfere



$O \in S \Rightarrow a^2+b^2+c^2=R^2 \quad (1)$

$A \in S \Rightarrow (2-a)^2+b^2+c^2=R^2 \quad (2)$

$B \in S \Rightarrow a^2+(5-b)^2+c^2=R^2 \quad (3)$

$C \in S \Rightarrow a^2+b^2+(3-c)^2=R^2 \quad (4)$

$$(1)-(2) = -4+4a=0$$
$$a=1$$
$$(1)-(3) = 10b-25=0$$
$$b=\frac{5}{2}$$
$$(1)-(4) = 6c-9=0$$
$$c=\frac{3}{2}$$

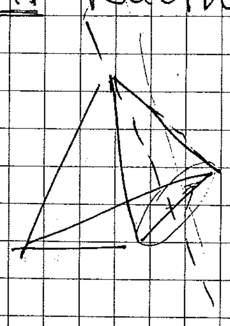
→ Uvrstimo u (1)

$$1+\frac{25}{4}+\frac{9}{4}=R^2$$

$$R^2=\frac{38}{4} \Rightarrow R=\frac{\sqrt{38}}{2}$$

$$S: (x-1)^2+\left(y-\frac{5}{2}\right)^2+\left(z-\frac{3}{2}\right)^2=\frac{38}{4}$$

II Način:



Simetrale stranica

→ Ravan prolazi kroz središte str AB

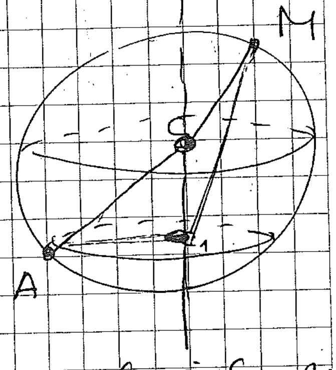
i ortogonalna je

→ presjek Centar

73) Odrediti j-nu sfere ako ona prolazi kroz krug

$$k: \begin{cases} (x-3)^2 + (y-4)^2 + z^2 = 36 \\ 4x + y - z - 9 = 0 \end{cases}$$

i kroz tačku $M(7,-3,1)$;



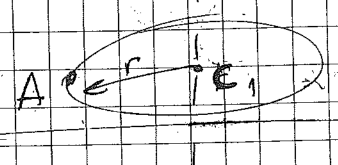
1) naći centar i poluprečnik kruga k → zadatak br. 70

C_1 i C_2 na istoj pravoj

• M

tačka sa sfere

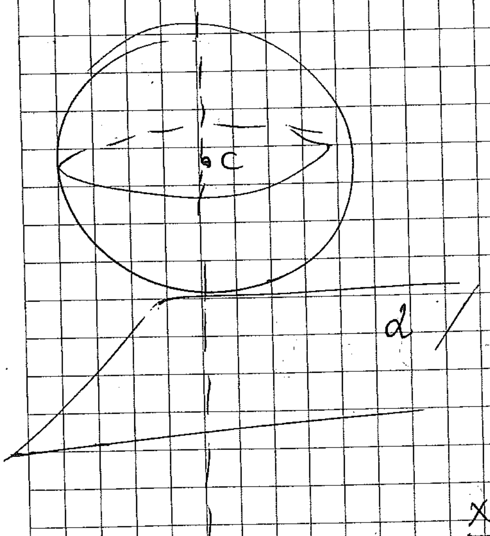
$$d(CM) = d(AC)$$



tortog.

74. Na sferi $(x-1)^2 + (y+2)^2 + (z-3)^2 = 5$ naći tačku M najbližu ravni α: $3x - 4z + 19 = 0$ i izračunati rastojanje tačke M od α.

$$C(1,-2,3) \qquad R = \sqrt{5}$$



p → prava kroz C i $\perp$ na α

$$\vec{s_p} = \lambda \cdot \vec{n_\alpha}$$

$$\vec{n_\alpha} = (3,0,-4)$$

$$\lambda = 1 \Rightarrow \vec{s_p} = (3,0,-4)$$

$$p: \frac{x-1}{3} = \frac{y+2}{0} = \frac{z-3}{-4}$$

$$p \cap (S) = \{M_1, M_2\}$$

$$p: \begin{cases} x = 1 + 3t \\ y = -2 + 0t \\ z = 3 - 4t \end{cases}$$

$M(x_0, y_0, z_0)$ $\alpha: ax + by + cz + d = 0$

$$d(M, \alpha) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$(\cancel{1} + 3t - \cancel{1})^2 + \underbrace{(-2 + 2)^2}_{0} + (\cancel{3} - 4t - \cancel{3})^2 = 5$$

$$9t^2 + 16t^2 = 5 \rightarrow 25t^2 - 5 = 0$$

$$5t^2 - 1 = 0 \rightarrow t^2 = \frac{1}{5}$$

$$t = \pm \frac{\sqrt{5}}{5}$$

$$x = 1 \pm 3 \cdot \frac{\sqrt{5}}{5}$$

$$y = -2$$

$$z = 3 \mp 4 \cdot \frac{\sqrt{5}}{5}$$

$$M_1\left(1 + 3\frac{\sqrt{5}}{5}, -2, 3 - 4\frac{\sqrt{5}}{5}\right)$$

$$M_2\left(1 - 3\frac{\sqrt{5}}{5}, -2, 3 + 4\frac{\sqrt{5}}{5}\right)$$

$d(M_1, \alpha) = 2 + \sqrt{5}$ $\rightarrow$ najbliža M_2 ravni α

$d(M_2, \alpha)$ $\sqrt{5} - 2$

---

15 Kroz pravu $p: \frac{x}{10} = \frac{y+2}{8} = \frac{z-1}{1}$ postaviti ravni koje dodiruju sferu:

$$S: (x+1)^2 + (y-3)^2 + (z+2)^2 = 29$$

$\rightarrow$ Odredimo pramen koji je određen pravom p

$$\frac{x}{10} = \frac{y+2}{8}$$

$$8x = 10y + 20 \rightarrow 4x - 5y - 10 = 0$$ ravan π_1

$$\frac{x}{10} = \frac{z-1}{1} \rightarrow x = 10z - 10$$

$$x - 10z + 10 = 0$$ ravan π_2

$\rightarrow$ Tražena ravan α pripada pramenu određenom ravnima π_1 i π_2. J-na pramena je:

$$4x - 5y - 10 + \lambda(x - 10z + 10) = 0$$

$(4+\lambda)x - 5y - 10\lambda z - 10 + 10\lambda = 0$

$d(C, \alpha) = R$

$C(-1, 3, -2)$; $R = \sqrt{29}$

$$\frac{|(4+\lambda)\cdot(-1) + 3\cdot(-5) + (-2)\cdot(-10\lambda) - (10+\lambda\cdot 10)|}{\sqrt{(4+\lambda)^2 + 25 + 100\lambda^2}} = \sqrt{29}$$

$\lambda_1 = -\frac{1}{4}$; $\lambda_2 = -\frac{2}{3}$

$\alpha_1: 3x - 4y + 2z - 10 = 0$

$\alpha_2: 2x - 3y + 4z - 10 = 0$

76) Odrediti jednačinu sfere koja dodiruje sve tri koordinatne ravni (leži u prvom oktantu) i dodiruje ravan

$\pi: x + 2y + 2z - 10 = 0$.

Napisati jednačinu ravni koja prolazi kroz dodirne tačke sfere i koordinatnih ravni.

~~S: (~~

~~R?~~ $R_{//}$ $S: (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$

$C(a, b, c)$

$d(C, xOy) = R \Rightarrow \frac{|c|}{1} = R \Rightarrow c = R$

S je u prvom oktantu → koord. centra su pozitivne

$d(C, xOz) = R \Rightarrow \frac{|b|}{1} = R \Rightarrow b = R$

$d(C, \underbrace{yOz}_{x=0}) = R \Rightarrow \frac{|a|}{1} = R \Rightarrow a = R$

$d(C, \Pi) = R \Rightarrow \frac{|a+2b+2c-10|}{\sqrt{1+4+4}} = R$

$\frac{|R+2R+2R-10|}{3} = R \rightarrow |5R-10| = 3R$

$5|R-2| = 3R \rightarrow |R-2| = \frac{3}{5}R$

1° $R-2 = \frac{3}{5}R$

$\frac{2}{5}R = 2 \quad /\cdot 5$

$2R = 10$

$\boxed{R=5}$

2° $R-2 = -\frac{3}{5}R$

$\frac{8}{5}R = 2 \quad /\cdot 5$

$8R = 10$

$\boxed{R = \frac{5}{4}}$

$S_1: (x-5)^2 + (y-5)^2 + (z-5)^2 = 25 \qquad S_2: (x-\frac{5}{4})^2 + (y-\frac{5}{4})^2 + (z-\frac{5}{4})^2 = \frac{25}{16}$

→ Imamo sferu i 3 koord. ravni

$\underbrace{A_1(5,5,0); \quad A_2(0,5,5), \quad A_3(5,0,5)}$

tačke u kojima sfera S_1 dodiruje koord. ravni

$\Pi_1(A_1, A_2, A_3)$ → ravan kroz 3 tačke

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$B_1 = (\frac{5}{4}, \frac{5}{4}, 0)$

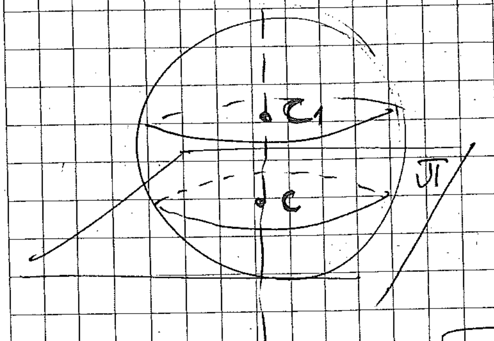
$B_2 = (\frac{5}{4}, 0, \frac{5}{4})$

$B_3 = (0, \frac{5}{4}, \frac{5}{4})$

tačke u kojima sfera S_2 dodiruje koord. ravni

$\Pi_2(B_1, B_2, B_3)$ → ravan

77. Sfera siječe ravan $\pi: x-2y-2z=1$ po krugu poluprečnika 3 čiji je centar u tački $C(5,1,1)$. Poluprečnik sfere je $R=5$. Kako glasi j-na sfere?



$p \to$ prava kroz C $\perp$ na π

$\vec{S_p} = \lambda \cdot \vec{n_\pi}$; $\lambda = 1$

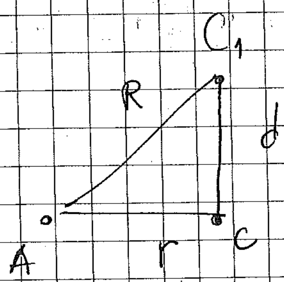
$\vec{S_p} = (1, -2, -2)$

$$p: \boxed{\frac{x-5}{1} = \frac{y-1}{-2} = \frac{z-1}{-2}}$$

$$p: \begin{cases} x = 5+t \\ y = 1-2t \\ z = 1-2t \end{cases} \quad C_1(5+t,\ 1-2t,\ 1-2t)$$

centar sfere

$A \to$ tačka sa kruga



$d = \sqrt{R^2 - r^2} = \sqrt{25-9} = 4$

$d(C, C_1) = 4$

$\sqrt{t^2 + 4t^2 + 4t^2} = 4$

rast. 2 tačke

$\sqrt{(5-(5+t))^2 + (1-(1-2t))^2 + (1-(1-2t))^2}$

$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

$t = \pm \frac{4}{3}$

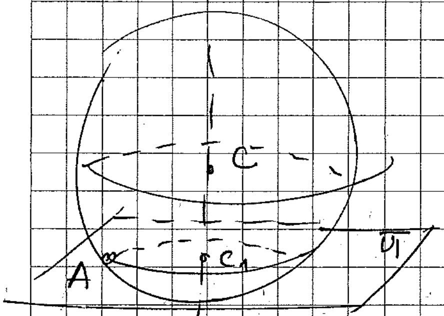
1° $C_1\left(\frac{19}{3}, -\frac{5}{3}, -\frac{5}{3}\right)$, $t=\frac{4}{3}$

$$S_1: \left(x-\frac{19}{3}\right)^2 + \left(y+\frac{5}{3}\right)^2 + \left(y+\frac{5}{3}\right)^2 = 25$$

2° $C_2\left(\frac{11}{3}, \frac{11}{3}, \frac{11}{3}\right)$

$$S_2: \left(x-\frac{11}{3}\right)^2 + \left(y-\frac{11}{3}\right)^2 + \left(z-\frac{11}{3}\right)^2 = 25$$

78 Napisati jednačinu ravni koja je normalna na pravu $p: \frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}$, siječe sferu $S: x^2+y^2+z^2=25$ po krugu poluprečnika 4 i siječe Oz osu u tački s pozitivnom aplikatom.



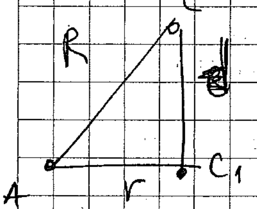
$C(0,0,0)$ → centar sfere

C_1 (centar kruga)

$r=4$ → poluprečnik kruga

$R=5$ → poluprečnik sfere

A → tačka sa kruga



$d = \sqrt{R^2-r^2} = \sqrt{25-16} = 3$

$d(C,C_1)=3$

$d(C,\pi)=3$

$d(C,\pi)=3 \Rightarrow \frac{|0+2\cdot 0+2\cdot 0+D|}{\sqrt{1+4+4}}$

$|D|=9$; $D<0 \Rightarrow D=-9$

π → ortogonalna na pravu p pa je $\pi: x+2y+2z-9=0$

$\boxed{\vec{n}_\pi = \lambda \vec{s}_p}$ za $\lambda=1 \Rightarrow \vec{n}_\pi = (1,2,2)$

→ J-na ravni π je $x+2y+2z+D=0$

$M(0,0,\lambda)$, $\lambda>0$ → presjek ravni π i ose Oz

$M\in\pi \Rightarrow 2\lambda + D = 0$, $D=-2\lambda$, $\lambda>0$ $\Rightarrow D<0$

CILINDAR

→ vodilja ili direktrisa

→ vektor pravca ili generatrisa

(79) Odrediti jednačinu cilindrične površi ako je njena direktrisa:

$$d: \begin{cases} x = y^2 + 2z^2 \\ x = 2z \end{cases}$$, a generatrisa normalna na ravan direktrise

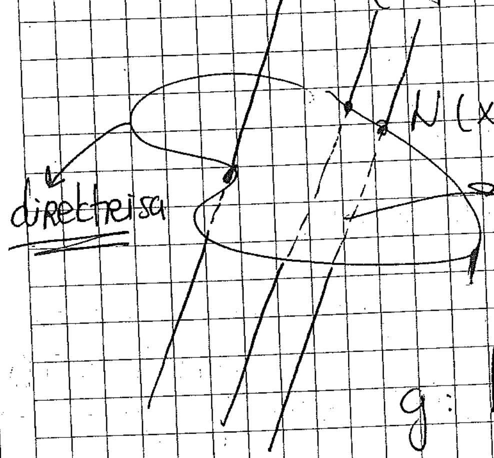
najčešće data kao presjek 2 površi

$\pi: \; x - 2z = 0 \rightarrow$ ravan direktrise

$\vec{s} \rightarrow$ vektor pravca generatrise g

$g \perp \pi \rightarrow \vec{s} = n \cdot \vec{n}_\pi$

$\vec{s} = (1, 0, -2)$, za $n = 1$



$\rightarrow N(x_0, y_0, z_0) \in d$

generatrisa $g: \; \dfrac{x - x_0}{1} = \dfrac{y - y_0}{0} = \dfrac{z - z_0}{-2}$

→ pređemo na parametarske j-ne

$$g: \begin{cases} x = x_0 + t \\ y = + y_0 \\ z = z_0 - 2t \end{cases} \Rightarrow \begin{cases} x_0 = x - t \\ y_0 = y \\ z_0 = z + 2t \end{cases}$$

?

$N \in d \Rightarrow$ koord. zad. i 1. i 2. j-nu $d \begin{cases} \mathrm{I} \\ \mathrm{II} \end{cases}$

→ iskoristimo 2. jednačinu direktrise

$x_0 = 2z_0 \rightarrow$ uvrstimo u g:

$x - t = 2(z + 2t)$

$5t = x - 2z \rightarrow \boxed{t = \dfrac{x - 2z}{5}}$

$$x_0 = x - \frac{x-2z}{5} = \frac{4x+2z}{5}$$

provjeriti!

$$y_0 = y$$

$$z_0 = z + 2 \cdot \frac{x-2z}{5} = \frac{2x+z}{5}$$

$N \in d \Rightarrow$ zad. i 2 jnu

$$x_0 = y_0^2 + 2z_0^2 \quad ?$$

→ uvrštavamo ovo što smo dobili

$$\left(\frac{4x+2z}{5}\right)^2 = y^2 + 2 \cdot \left(\frac{2x+z}{5}\right)^2 \quad ?$$

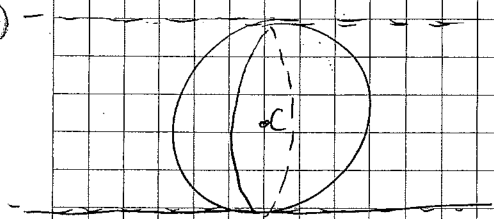
80. Sfera $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$ je osvijetljena zracima koji su paralelni pravoj p:

p: $x = y = z$; Naći oblik sjenke na ravan xOy

R//

1) Doći do cilindra

2) presjeći sa xOy osom → to je sjenka



→ Zraci koji tangiraju sferu formiraju cilindričnu površ

$\vec{s} = (1,1,1)$ → vektor pravca generatrise

~~na osnovu toga~~

na osnovu pravca prave p → $\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

α: Ravan koja sadrži centar sfere i ortogonalna je na zrake tj na p

$\vec{n_\alpha} = \lambda \cdot \vec{s_p}$; za $\lambda = 1 \Rightarrow \vec{n_\alpha} = (1,1,1)$

ravan α → sadrži $C(1,1,1)$ i ⊥ na p ⇒ $\vec{n}_\alpha = (1,1,1)$

$$\alpha: \quad 1\cdot(x-1) + 1\cdot(y-1) + 1(z-1) = 0$$

$$\alpha: \quad x + y + z - 3 = 0$$

→ direktrisa cilindra je presjek sfere i ravni α (krug)

$$d: \begin{cases} (x-1)^2 + (y-1)^2 + (z-1)^2 = 1 \\ x + y + z - 3 = 0 \end{cases}$$

$N(x_0, y_0, z_0) \in d$

$$g: \frac{x-x_0}{1} = \frac{y-y_0}{1} = \frac{z-z_0}{1}$$

vektor pravca isti kao p

$$g: \begin{cases} x = x_0 + t \Rightarrow x_0 = x - t \\ y = y_0 + t \Rightarrow y_0 = y - t \\ z = z_0 + t \Rightarrow z_0 = z - t \end{cases}$$

$N \in d \Rightarrow x - t + y - t + z - t - 3 = 0$

$$x_0 + y_0 + z_0 - 3 = 0$$

$$3t = x + y + z - 3$$

$$t = \frac{x+y+z}{3} - 1$$

$$x_0 = x - \frac{x+y+z}{3} - 1 = \frac{2x - y - z}{3} + 1$$

$$y_0 = y - \frac{x+y+z}{3} - 1 = \frac{-x + 2y - z}{3} + 1$$

$$z_0 = z - \frac{x+y+z}{3} - 1 = \frac{-x - y + 2z}{3} + 1$$

Ne d $\Rightarrow$ zadovoljava j-nu sfere

$$(x_0-1)^2+(y_0-1)^2+(z_0-1)^2=0$$

$\rightarrow$ uvrstimo x_0, y_0 i z_0

$$\left(\frac{2x-y-z}{3}-1+1\right)^2+\left(\frac{-x+2y-z}{3}\right)^2+\left(\frac{-x-y+2z}{3}\right)^2=1$$

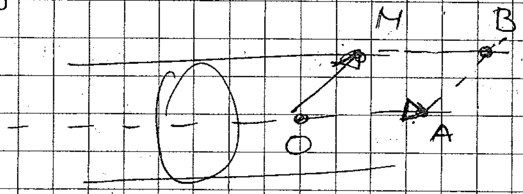
J-na cilindrične površi

Sjenka $\rightarrow$ $xOy \; z=0$

$$\begin{cases} z=0 \\ \left(\frac{2x-y-z}{3}\right)^2+\left(\frac{-x+2y-z}{3}\right)^2+\left(\frac{-x-y+2z}{3}\right)^2=1 \end{cases}$$

81. Odrediti jednačinu kružnog cilindra kojem je osa prava $p: \frac{x}{1}=\frac{y}{5}=\frac{z}{2}$, a poluprečnik 3.

J-ne koje mora zadovoljavati svaka tačka:



$O(0,0,0)$

$\vec{s_p}=(1,5,2)$

$A\in p$; $\overrightarrow{OA}=\vec{s_p}$

$M(x,y,z)\rightarrow$ tačka cilindra

nad $\overrightarrow{OA}$ i $\overrightarrow{OM}$ je paralelogram $OAMB$

$P\rightarrow$ površina paralelograma

$$\left.\begin{aligned} P&=|\overrightarrow{OA}\times\overrightarrow{OM}|=|\vec{s_p}\times\overrightarrow{OM}| \\ P&=|\overrightarrow{OA}|\cdot r=|\vec{s_p}|\, r \end{aligned}\right\}$$

$$\Longrightarrow |\vec{s_p}\times\overrightarrow{OM}|=|\vec{s_p}|\cdot r$$

$$\vec{s_p}\times\overrightarrow{OM}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & 2 \\ x & y & z \end{vmatrix}$$

$\vec{S_p} \times \vec{OM} = (5z-2y)\vec{i} - (z-2x)\vec{j} + (y-5x)\vec{k}$

$\vec{S_p} \times \vec{OM} = (5z-2y,\ 2x-z,\ y-5x)$

$* \Rightarrow \sqrt{(5z-2y)^2 + (2x-z)^2 + (y-5x)^2} = \sqrt{30}\cdot 3 \quad /^2$

$\sqrt{25+4+1}$

$(1,5,2)$

$(5z-2y)^2 + (2x-z)^2 + (y-5x)^2 = 270$

$\boxed{|\vec{S_p} \times \vec{OM}| = |\vec{S_p}|\cdot r}$

$\underbrace{(5z-2y)^2 + (2x-z)^2 + (y-5x)^2 = 270}$

J-na cilindrične površi

→ Izvodnica → svaka prava sa cilindra

PAZITI → ako su date 2 izvodnice udaljenost nije 2R !

82 Date su sfere:

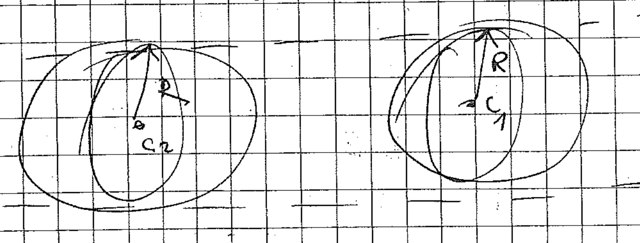
$S_1: x^2+y^2+z^2=25$

$S_2: (x-2)^2+(y-1)^2+z^2=25$

a) Napisati j-nu cilindra opisanog oko sfera S_1 i S_2

b) Naći presek dobijenog cilindra i ravni xOy

2 sfere istog poluprečnika, opisati cilindar



prava $p(C_1, C_2)$ je osa cilindra;

$C_1\ (0,0,0)$; $C_2\ (2,1,0)$

$\vec{s} = \overrightarrow{C_1C_2} = (2,1,0)$ → vektor pravca generatrise

α → ravan kroz C_1 i $\perp$ na p

$\vec{n}_\alpha = (2,1,0)$

$\alpha: \quad 2(x-0) + 1\cdot(y-0) + 0\cdot(z-0) = 0$

$\alpha: \quad 2x + y = 0$

$$d = \begin{cases} x^2 + y^2 + z^2 = 25 & \rightarrow \text{sfera} \\ 2x + y = 0 & \rightarrow \text{dobijena ravan} \end{cases}$$

u presjeku krug direktrisa

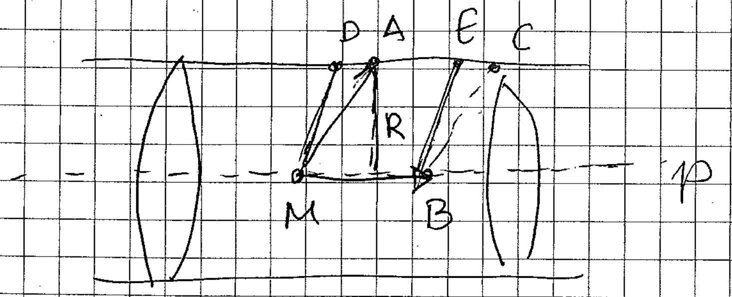
→ zad. imali osvijetljenu sferu slično

83 Napisati jednačinu kružnog cilindra koji prolazi kroz tačku $A(2,-1,1)$ a osa mu je prava p:

$$p: \begin{cases} x = 3t + \cancel{x} + 1 \\ y = -2t - 2 \\ z = t + 2 \end{cases}$$

→ slično 81. samo što ovdje nemamo poluprečnik

$X = 3t + 1$



$M(1,-2,2) \in p$

$\vec{s}_p = (3,-2,1)$

$$p: \quad \frac{x-1}{3} = \frac{y+2}{-2} = \frac{y-2}{1}$$

$B \in p$, $\overrightarrow{MB} = \overrightarrow{s_p}$

Nad $\overrightarrow{MB}$ i $\overrightarrow{MA}$ paralelogram MBCA

$$P_{\square MBCA} = |\overrightarrow{MB} \times \overrightarrow{MA}| = |\overrightarrow{s_p} \times \overrightarrow{MA}|$$

$$P_{\square MBCA} = |\overrightarrow{MB}| \cdot r = |\overrightarrow{s_p}| \cdot r$$

$\rightarrow$ dobijemo r i radimo kao prije (1. način)

ili $\rightarrow$ 2. način

uzmimo tačku $D(x, y, z)$ $\rightarrow$ tačka cilindra

Nad $\overrightarrow{MB}$ i $\overrightarrow{MD}$ paralelogram MBED

$$P_{\square MBED} = |\overrightarrow{MB} \times \overrightarrow{MD}| = |\overrightarrow{s_p} \times \overrightarrow{MD}| \quad (2)$$

$$P_{\square MBCA} = |\overrightarrow{MB} \times \overrightarrow{MA}| = |\overrightarrow{s_p} \times \overrightarrow{MA}| \quad (1)$$

iz (1) i (2)

$$|\overrightarrow{s_p} \times \overrightarrow{MA}| = |\overrightarrow{s_p} \times \overrightarrow{MD}|$$

$\overrightarrow{MA} = (1, 1, -1)$ $\quad$ $\overrightarrow{s_p} = (3, -2, 1)$

$\overrightarrow{MD} = (x-1, y+2, z-1)$

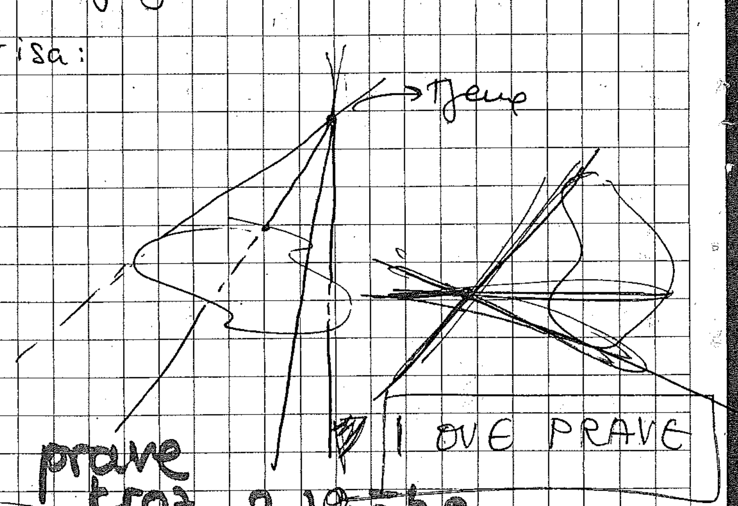
uvrstimo, računamo vektorski proizvod

$\rightarrow$ ili r kao prethodni zadatak

KONUS

04. Odrediti j-nu konusa čiji je vrh tačka $S(0,-a,0)$ a direktrisa:

$$d: \begin{cases} x^2+y^2+z^2=a^2 \\ y+z=a \end{cases}$$



$N(x_0,y_0,z_0) \in d$ j-na prave kroz 2 tačke

$$g: \frac{x-0}{x_0-0} = \frac{y-(-a)}{y_0-(-a)} = \frac{z-0}{z_0-0}$$ → na osnovu čega ?

$$g: \frac{x}{x_0} = \frac{y+a}{y_0+a} = \frac{z}{z_0}$$

$$\frac{x - \text{tačka } 1}{\text{tačka } 2 - \text{tačka } 1}$$

$$g: \begin{cases} x = x_0 \cdot t & \Rightarrow x_0 = \frac{x}{t} \\ y = -a + (y_0+a)\cdot t & \Rightarrow y_0 = \frac{y+a-at}{t} \\ z = z_0 \cdot t & \Rightarrow z_0 = \frac{z}{t} \end{cases}$$

Komentar: ako je $t=0$ $x=0$, $y=-a$, $z=0$ → tačka $S(0,-a,0)$ tjeme konusa

$N \in d \Rightarrow y_0 + z_0 = a$

$$\frac{x}{t} + \frac{y+a-at}{t} = a$$

$$\frac{x+y+a-}{t} - a = a$$

$$\boxed{t = \frac{x+y+a}{2a}}$$

$$x_0 = \frac{2ax}{x+y+a}$$

$$y_0 = \frac{a(y-z+a)}{y+z+a}$$

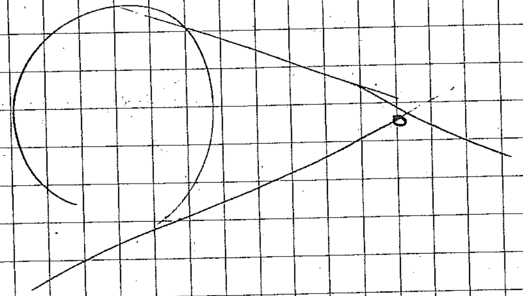
$$z_0 = \frac{2az}{y+z+a}$$

→ $N \in d \Rightarrow$ zadovoljava j-nu sfere

$\Rightarrow$ uvrštavamo dobijeno

$$\left(\frac{2ax}{x+y+a}\right)^2 + \left(\frac{a(y-z+a)}{y+z+a}\right)^2 + \left(\frac{2az}{y+z+a}\right)^2 = a^2$$

85. Sfera $x^2+y^2+z^2=9$ osvijetljena je svjetlošću čiji se izvor nalazi u tački $M(5,0,0)$. Naći oblik sjenke u ravni yOz.



→ Zraci koji tangiraju sferu formiraju cilindričnu površ; neka je ℓ takav zrak i $\vec{s}=(m,n,p)$ njegov vektor pravca.

$$\ell: \frac{x-5}{m}=\frac{y}{n}=\frac{z}{p}$$

→ Nađimo $\ell \cap S$

$$\ell: \begin{cases} x=5+mt \\ y=nt \\ z=pt \end{cases}$$

uvrstimo u jednačinu sfere:

$$(5+mt)^2+n^2t^2+p^2t^2=9$$

$$25+10mt+m^2t^2+n^2t^2+p^2t^2=9$$

$$t^2(m^2+n^2+p^2)+10mt+16=0 \quad (*)$$

mora imati samo jedno rješenje,

$$\boxed{\Rightarrow D=0}$$

$$100m^2-4(m^2+n^2+p^2)\cdot 16=0$$

$$9m^2-16n^2-16p^2=0 \quad (**)$$

→ vektor pravca proizvoljnog zraka koji tangira sferu mora zadovoljavati jednačinu (**)

→ Neka je $A(x,y,z)$ proizvoljna tačka konusa

→ **vektor pravca zraka određenog tačkama M i A je $\overrightarrow{MA}=(x-5,y,z)$**

Po zadovoljinju (**)

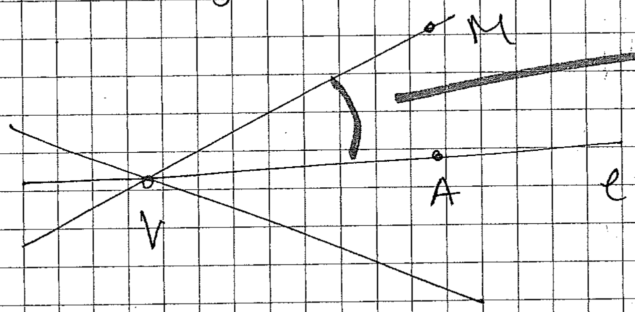
$$\boxed{9(x-5)^2 - 16y^2 - 16z^2 = 0} \rightarrow \text{J-na konusa}$$

sjecka: $x = 0$

$$9 \cdot 25 = 16y^2 + 16z^2$$

sjecka $\begin{cases} x = 0 \\ 9(x-5)^2 - 16y^2 - 16z^2 = 0 \end{cases}$

86 Prava $\ell: \dfrac{x-2}{2} = \dfrac{y+1}{-2} = \dfrac{z+1}{-1}$ je osa kružnog konusa sa vrhom u yOz ravni. Tačka $M(1,1,-1)$ pripada konusu. Naći j-nu konusa



treba ugao otvora tj. polovina

V - vrh konusa

$$\ell \cap yOz = \{V\}$$

$\begin{cases} x = 2 + 2t \\ y = -1 - 2t \\ z = -1 - t \end{cases}$

za $x = 0$

$\Rightarrow 2t = -2$

$\boxed{t = -1}$

$y = -1 + 2 = 1$

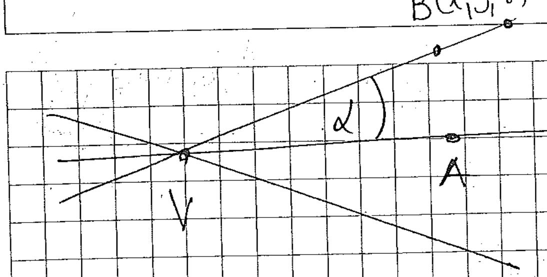
$z = -1 + 1 = 0$

$$\boxed{V(0,1,0)}$$

$A \in \ell$, $\vec{VA} = \vec{s_\ell}$

$\vec{s_\ell} = (2, -2, -1)$

$M(1,1,-1)$ $V(0,1,0)$



$A \in l$; $\vec{VA} = \vec{s_l} = (2,-2,-1)$

$\alpha \rightarrow$ ugao između izvodnice i ose

$$\alpha = \sphericalangle(\vec{s_l}, \vec{VM}), \quad \vec{VM} = (1,0,-1)$$

$$\cos\alpha = \frac{\vec{s_l}\cdot\vec{VM}}{|\vec{s_l}|\cdot|\vec{VM}|} = \frac{2+0+1}{\sqrt{4+4+1}\cdot\sqrt{2}} = \frac{3}{3\sqrt2} = \frac{1}{\sqrt2} = \frac{2}{\sqrt2}$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

$B(x,y,z) \rightarrow$ tačka konusa

$\alpha \rightarrow$ ugao između $\vec{VB}$ i $\vec{s_l}$ $(-\vec{s_l}$

$$\cos\alpha = \pm\frac{\vec{VB}\cdot\vec{s_l}}{|\vec{VB}|\cdot|\vec{s_l}|}$$

$$\frac{\sqrt2}{2} = \pm\frac{2x+(-2)(y-1)+z(-1)}{\sqrt{x^2+(y-1)^2+z^2}\cdot 3}$$

$$2(2x-2y-z+2)^2 = 9x^2+9(y-1)^2+9z^2$$

→ Većina zadataka ugao otvora!

(87) Napisati jednačinu konusa koji je opisan oko sfera

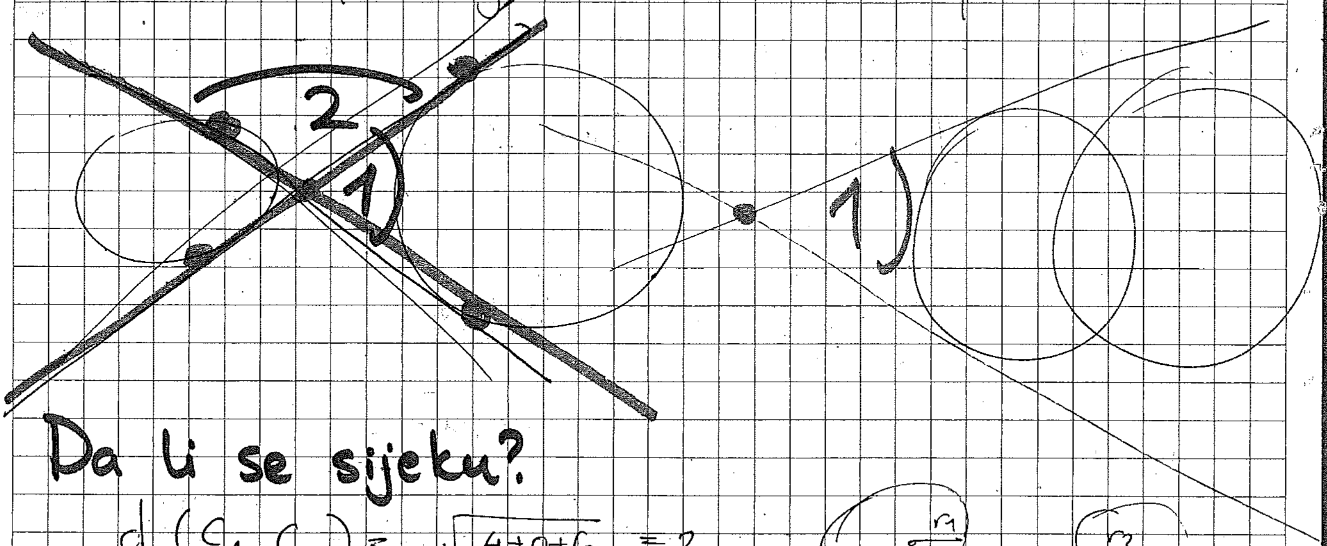
$S_1: x^2+y^2+z^2=1$

$S_2: (x-2)^2+y^2+z^2=4$

$C_1(0,0,0)$ $R_1=1$

$C_2(2,0,0)$, $R_2=2$

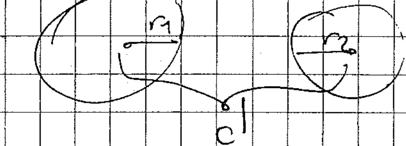
→ uočimo kakav je odnos između sfera



Da li se sijeku?

$d(C_1, C_2) = \sqrt{4+0+6} = 2$

kada se sijeku $d < r_1 + r_2$



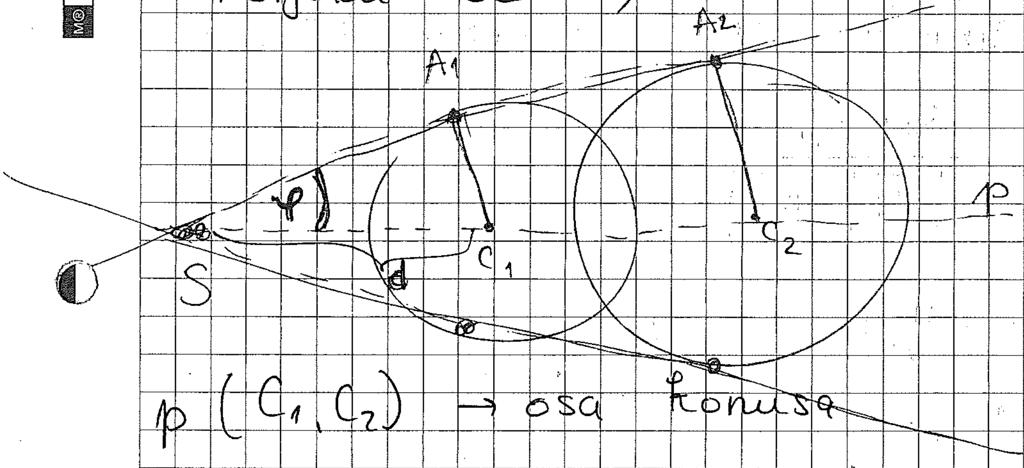
$2 < 1+2$

$d < R_1 + R_2 \Rightarrow$ sfere se sijeku

(Mogli smo uočiti da $C_1 \in S_2$)

$C_1(0,0,0)$
$C_2(2,0,0)$

→ sijeku se ⇒



$p(C_1, C_2)$ → osa konusa

$$\frac{x-0}{2-0} = \frac{x-0}{0-0} = \frac{z-0}{0-0}$$ (0x osa)

$\triangle SC_1A_1 \sim \triangle SC_2A_2$ ($\sphericalangle C_1A_1S = \sphericalangle C_2A_2S = \frac{\pi}{2}$, $\sphericalangle C_1SA_1 = \sphericalangle C_2SA_2 = \varphi$

$d(S, C_1) = d$

$$\frac{SC_1}{C_1A_1} = \frac{SC_2}{C_2A_2} \Rightarrow \frac{d}{1} = \frac{d+2}{2}$$

→ udaljenost C_1 i C_2

$2d = d+2$

$d = 2$

$S \in p \,(Ox\text{-osa}) \rightarrow S(a,0,0)$

$S(a,0,0) \qquad C_1(0,0,0) \qquad C_2(2,0,0)$

$\rightarrow |SC| = \sqrt{a^2} \Rightarrow a = \pm 2 \rightarrow \boxed{a=2}$

$\rightarrow$ treba ugao ΔSC_1A_1: $\quad \sin\varphi = \dfrac{C_1A_1}{SC_1}$

$\sin\varphi = \dfrac{1}{2} \qquad \boxed{\varphi = \dfrac{\pi}{6}}$

Napomena:

$\cos\alpha = \sqrt{1-\sin^2\alpha} \rightarrow$ on nam treba

Neka je $M(x,y,z)$ tačka konusa

$\varphi \rightarrow$ ugao između $\vec{SM}$ i vektora $\vec{S_p}$ $(-\vec{S_p})$

$$\cos\varphi = \pm \frac{\vec{SM}\cdot\vec{S_p}}{|\vec{SM}|\,|\vec{S_p}|}$$

$$\vec{SM} = (x+2, y, z)$$

$$\vec{S_p} = (1,0,0)$$

$$\frac{\sqrt{3}}{2} = \pm \frac{x+2}{\sqrt{(x+2)^2+y^2+z^2}}$$

$$4(x+2)^2 = 3(x+2)^2 + 3y^2 + 3z^2$$

$$\boxed{(x+2)^2 - 3y^2 - 3z^2 = 0}$$

Jednačina konusa

88 Odrediti j-nu konusa čiji je vrh centar kruga

$$k: \begin{cases} x^2 + y^2 + z^2 - 4x - 4y + 6z - 19 = 0 \\ 3x + y - z = 0 \end{cases}$$

a direktrisa ortogonalna projekcija kruga (k) na ravan $y+10=0$

1. korak → krug → presek sfere i ravni
2. direktrisa ortogonalna projekcija → cilindar; krug direktrisa cilindra;

vektor pravca ortogonalan na ravan $\boxed{y+10=0}$

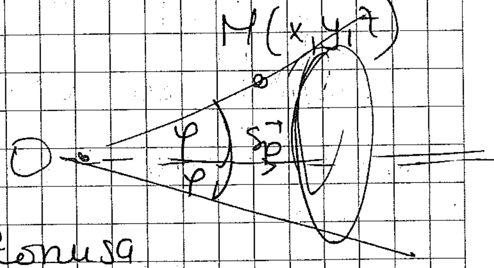
→ kroz svaku tačku kruga normala na ravan

→ u preseku sa tom ravni → projekcija;

$$\underbrace{d: \begin{cases} \text{cilindar} \\ y+10=0 \end{cases}}_{\text{direktrisa konusa}}$$

$$\boxed{\begin{matrix}\text{vrh} \\ + \\ \text{direktrisa}\end{matrix}}$$

89) Napisati j-nu konusa kojem je osa prava $p: \frac{x}{2} = \frac{y}{3} = \frac{z}{6}$, vrh u koord. početku a ugao otvora pri vrhu $2\varphi = 60^\circ$



$\vec{s_p} = (2, 3, 6)$

M ∈ konusa → proizvoljna tačka sa konusa

$\measuredangle(\vec{OM}, \vec{s_p}) = 30^\circ$

$\vec{OM} = (x, y, z)$

$x-0,\ y-0,\ z-0$

$\cos\measuredangle(\vec{OM}, \vec{s_p}) = \cos\alpha = \frac{\sqrt{3}}{2}$

$$\pm \frac{\vec{OM}\cdot\vec{s_p}}{|\vec{OM}|\cdot|\vec{s_p}|} = \frac{\sqrt{3}}{2}$$

$$\pm \frac{2x+3y+6z}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{4+9+36}} = \frac{\sqrt{3}}{2} \quad /^2$$

$$\frac{(2x+3y+6z)^2}{(x^2+y^2+z^2)\cdot 7} = \frac{\sqrt{3}}{2}$$

$$2(2x+3y+6z)^2 = 21(x^2+y^2+z^2)$$